講義3：意思決定と不確実性

上級ミクロ経済学 — 財務省理論研修

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Readings | 参考文献

テキスト

- *JR: Ch1; Ch2
- 神取：1 章; 6 章
- 尾山・安田: 9 章

配布資料

- *Laffont, The Economics of Uncertainty and Information, 1989: Ch2

関連図書

- ギルボア 『合理的選択』みすず書房, 2013.
To construct a model of individual choice, the notion of preferences (選好) plays a central role in economic theory, which specifies the form of consistency or inconsistency in the person’s choices.

We view preferences as the mental attitude of an individual toward alternatives independent of any actual choice.

- We require only that the individual make binary (二項の) comparisons, that is, that she only examine two choice alternatives at a time and make a decision regarding those two.
- For each pair of alternatives in the choice set $X$, the description of preferences should provide an answer to the question of how the agent compares the two alternatives.
- Considering questionnaire (アンケート) $R$, we formulate the consistency requirements necessary to make the responses “preferences”.
Definition 1

Preferences \((R)\) on a set \(X\) is a binary relation \(\succeq\) on \(X\) satisfying the following two axioms.

**Axiom 1: Completeness (完備性)**

For any \(x, y \in X\), \(x \succeq y\) or \(y \succeq x\).

**Axiom 2: Transitivity (推移性):**

For any \(x, y, z \in X\), if \(x \succeq y\) and \(y \succeq z\), then \(x \succeq z\).
Remarks on the Axioms | 公理に関する注意

- **Completeness** formalizes the notion that the individual can make comparisons, that is, that she has the ability to discriminate and the necessary knowledge to evaluate alternatives. It says the individual can examine *any* two distinct alternatives.

- **Transitivity** gives a very particular form to the requirement that the individual choices be *consistent*. Although we require only that she be capable of comparing two alternatives at a time, the axiom of transitivity requires that those pairwise comparisons be linked together in a consistent way.

Rm The money pump argument when transitivity is violated.

→ 推移性を満たさない個人からはお金をいくらでも吸い取ることができる！
Questionnaire $P$ | アンケート $P$

For all distinct $x$ and $y$ in the set $X$. How do you compare $x$ and $y$?
Tick one and only one of the following three options:

1. I prefer $x$ to $y$, or $x$ is strictly preferred (強く選好される) to $y$: $x \succ y$
2. I prefer $y$ to $x$, or $y$ is strictly preferred to $x$: $y \succ x$
3. I am indifferent, or $x$ is indifferent (無差別である) to $y$: $x \sim y$

Note that we implicitly assume that the elements in $X$ are all comparable, and ignore the intensity of preferences.

A legal answer to the questionnaire $P$ can be formulated as a function $f$ which assigns to any pair $(x, y)$ of distinct elements in $X$ exactly one of the three values: $x \succ y$, $y \succ x$ or $x \sim y$. That is,

$$f(x, y) = \begin{cases} 
  x \succ y \\
  y \succ x \\
  x \sim y 
\end{cases}.$$
Preferences are characterized by axioms (公理) that are intended to give formal mathematical expression to fundamental aspects of choice behavior and attitudes toward the objects of choice.

The following basic axioms are (almost) always imposed.

Definition 2

Preferences \((P)\) on a set \(X\) are a function \(f\) that assigns to any pair \((x, y)\) of distinct elements in \(X\) exactly one of the three values: \(x \succ y\), \(y \succ x\) or \(x \sim y\) so that for any three different elements \(x, y\) and \(z\) in \(X\), the following two properties hold:

1. No order effect: \(f(x, y) = f(y, x)\).
2. Transitivity:
   1. if \(f(x, y) = x \succ y\) and \(f(y, z) = y \succ z\), then \(f(x, z) = x \succ z\), and
   2. if \(f(x, y) = x \sim y\) and \(f(y, z) = y \sim z\), then \(f(x, z) = x \sim z\).
The first property requires the answer to $P(x, y)$ being identical to the answer to $P(y, x)$, and the second requires that the answer to $P(x, y)$ and $P(y, z)$ are consistent with the answer to $P(x, z)$ in a particular way.

**Ex** Non-preference relation

For any $x, y \in \mathbb{R}$, $f(x, y) = f(y, x) = x \succ y$ if $x \geq y + 1$ and $f(x, y) = x \sim y$ if $|x - y| < 1$. This is not a preference relation since transitivity is violated. For instance, suppose $x = 1, y = 1.8, z = 2.6$. Then,

$$f(x, y) = x \sim y \text{ and } f(y, z) = y \sim z,$$

but $f(x, z) = z \succ x$,

which violates transitivity (2-2).
We can translate one formulation of preferences to another by the following mapping (bijection). Note that completeness guarantees “$x \nless y$ and $y \nleq x$” never happen.

- $f(x, y) = x \succ y \iff x \succeq y$ and $y \nleq x$.
- $f(x, y) = y \succ x \iff y \succeq x$ and $x \nleq y$.
- $f(x, y) = x \sim y \iff x \succeq y$ and $y \succeq x$.

In our lectures, we take the second definition, i.e., preference ($R$), and denote $x \succ y$ when both $x \succeq y$ and $y \nleq x$, and $x \sim y$, when $x \succeq y$ and $y \succeq x$.

**Definition 3**

A preference ($R$) is called a preference relation (選好関係).
**Definition 4 (Review)**

Function $U : X \rightarrow \mathbb{R}$ represents the preference $\succeq$ if for all $x$ and $y \in X$, $x \succeq y$ if and only if $U(x) \geq U(y)$. If the function $U$ represents the preference relation $\succeq$, we refer to it as a **utility function** and we say that $\succeq$ has a **utility representation** (効用表現).

**Q** Under what conditions do utility representations exist?

**Theorem 1**

*If $\succeq$ is a preference relation on a finite set $X$, then $\succeq$ has a utility representation with values being natural numbers.*

**Proof.**

There is a minimal (resp. maximal) element (an element $a \in X$ is minimal (resp. maximal) if $a \preceq x$ (resp. $a \succeq x$) for any $x \in X$) in any finite set $A \subset X$. We can construct a sequence of sets from the minimal to the maximal and can assign natural numbers according to their ordering. \(\square\)
Important difference between choice (demand) and preferences or utility is that the former is *observable* while the latter *cannot* be. We may want to develop the theory which is based on the observable choice behaviors, not on preferences or utility.

- We say that the preferences $\succeq$ (fully) **rationalize** the demand function $x$ if for any $(p, \omega)$ the bundle $x(p, \omega)$ is the unique $\succeq$ best bundle within $B(p, \omega)$.

- We say that $a$ is **revealed** to be better than $b$, if there is $(p, \omega)$ so that both $a$ and $b$ are in $B(p, \omega)$ and $a = x(p, \omega)$.

What are general conditions guaranteeing that a demand function $x(p, \omega)$ can be rationalized?

→ Present two axioms of **revealed preferences** (顯示選好).
Definition 5 (Weak Axiom)

The weak axiom of revealed preferences (WA) is a property of choice function which says that it is impossible that $a$ be revealed to be better than $b$ and $b$ be revealed to be better than $a$. That is,

$$\text{if } px(p', \omega') \leq \omega \text{ and } x(p, \omega) \neq x(p', \omega'),$$
then $$p'x(p, \omega) > \omega'.$$

Figure 2.3 (see JR, pp.92)

Note that any choice function rationalized by some preference relation must satisfy WA.
Theorem 2

Let \( x(p, \omega) \) be a choice function satisfying Walras’s Law and WA. Then,

1. \( x(\cdot) \) is homogeneous of degree zero, and
2. if \( \omega' = p'x(p, \omega) \), then either \( x(p', \omega') = x(p, \omega) \) or
   \((p' - p)(x(p', \omega') - x(p, \omega)) < 0\).

Proof.

The proof for 1 is left for the assignment. Assume that \( x(p', \omega') \neq x(p, \omega) \). By Walras’s Law and the assumption that \( \omega' = p'x(p, \omega) \):

\[
(p' - p)(x(p', \omega') - x(p, \omega))
= p'x(p', \omega') - p'x(p, \omega) - px(p', \omega') + px(p, \omega)
= \omega' - \omega' - px(p', \omega') + \omega
= \omega - px(p', \omega').
\]

By WA the right hand side is less than 0. \( \square \)
The previous theorem implies that the compensated (Hicksian) demand function \( y(p') = x(p', p'x(p, \omega)) \) satisfies the law of demand (需要法則), that is, \( y_k \) is decreasing in \( p_k \).

WA is not a sufficient condition for extending the binary relation \( \succeq \) (defined from the choice function) into a complete and transitive relation. The following stronger condition than WA is known to be necessary and sufficient.

**Definition 6 (Strong Axiom)**

Choice function satisfies the **strong axiom of revealed preferences (SA)** if for every sequence of distinct bundles \( x^0, x^1, \ldots, x^k \), where \( x^0 \) is revealed preferred to \( x^1 \), and \( x^1 \) is revealed preferred to \( x^2 \), ..., and \( x^{k-1} \) is revealed preferred to \( x^k \), it is not the case that \( x^k \) is revealed preferred to \( x^0 \).
We have so far not distinguished between individual’s actions (行動) and consequences (帰結), but many choices made by agents take place under conditions of uncertainty (不確実性).

This lecture discusses such a decision under uncertainty, i.e., an environment in which the correspondence between actions and consequences is not deterministic (確定的) but stochastic (確率的).

- To discuss a decision under uncertainty, we extend the domain of choice functions. The choice of an action is viewed as choosing a “lottery” (くじ) where the prizes are the consequences.
- An implicit assumption is that the decision maker does not care about the nature of the random factors but only about the distribution (分布) of consequences.
We consider preferences and choices over the set of “lotteries.”

- Let $S$ be a set of consequences or prizes (賞). We assume that $S$ is a finite set and the number of its elements ($= |S|$) is $S$.
- A lottery $p$ is a function that assigns a nonnegative number to each prize $s$, where $\sum_{s \in S} p(s) = 1$ (here $p(s)$ is the objective probability (客観確率) of obtaining the prize $s$ given the lottery $p$).
- Let $\alpha \circ x \oplus (1 - \alpha) \circ y$ denote the lottery in which the prize $x$ is realized with probability $\alpha$ and the prize $y$ with $1 - \alpha$.
- Denote by $L(S)$ the (infinite) space containing all lotteries with prizes in $S$. That is, $\{x \in \mathbb{R}_+^S | \sum x_s = 1\}$.
- We will discuss preferences over $L(S)$. 
We impose the following three assumptions on the lotteries.

1. $1 \oplus (1 - 1) \odot y \sim x$: Getting a prize with probability one is the same as getting the prize for certain.

2. $\alpha \oplus (1 - \alpha) \odot y \sim (1 - \alpha) \odot y \oplus \alpha \odot x$: The consumer does not care about the order in which the lottery is described.

3. $\beta \odot x \oplus (1 - \alpha) \odot y \oplus (1 - \beta) \odot y \sim (\beta \alpha) \odot x \oplus (1 - \beta \alpha) \odot y$: A consumer’s perception of a lottery depends only on the net probabilities of receiving the various prizes.

The first two assumptions appear to be innocuous.

The third assumption sometimes called “reduction of compound lotteries” is somewhat suspect.

There is some evidence to suggest that consumers treat compound lotteries different than one-shot lotteries.
The most primitive way to evaluate a lottery is to calculate its mathematical expectation, i.e., \( E[p] = \sum_{s \in S} p(s)s \).

Daniel Bernoulli first doubt this approach in the 18th century when he examined the St Petersburg paradox.

**Ex** St Petersburg Paradox

A fair coin is tossed until it shows heads for the first time. If the first head appears on the \( k \)-th trial, a player wins \( \$2^k \). How much are you willing to pay to participate in this lottery?

**Rm** The expected value of the lottery is infinite:

\[
\frac{2}{2} + \frac{2^2}{2^2} + \frac{2^3}{2^3} + \cdots = 1 + 1 + 1 + \cdots = \infty.
\]
The St Petersburg paradox shows that maximizing your dollar expectation may not always be a good idea. It suggests that an agent in risky situation might want to maximize the expectation of some “utility function” with decreasing marginal utility:

\[ E[u(x)] = u(2) \frac{1}{2} + u(4) \frac{1}{4} + u(8) \frac{1}{8} + \cdots, \]

which can be a finite number.

**Q** Under what kinds of conditions does a decision maker maximizes the expectation of some “utility function”?

**Rm** By utility theory, we know that for any preference relation defined on the space of lotteries that satisfies continuity, there is a utility representation \( U : L(S) \rightarrow \mathbb{R} \), continuous in the probabilities, such that \( p \succeq q \) if and only if \( U(p) \geq U(q) \).
We will use the following two axioms to isolate a family of preference relations which have a representation by a more structured utility function.

- **Independence Axiom (I, 独立性公理):** For any \( p, q, r \in L(S) \) and any \( \alpha \in (0, 1) \),

\[
p \succeq q \iff \alpha \circ p \oplus (1 - \alpha) \circ r \succeq \alpha \circ q \oplus (1 - \alpha) \circ r.
\]

- **Continuity Axiom (C, 連続性公理):** If \( p \succ q \succ r \), then there exists \( \alpha \in (0, 1) \) such that

\[
q \sim [\alpha \circ p \oplus (1 - \alpha) \circ r].
\]

**Theorem 3**

Let \( \succcurlyeq \) be a preference relation over \( L(S) \) satisfying the I and C. There are numbers \( (v(s))_{s \in S} \) such that

\[
p \succeq q \iff U(p) = \sum_{s \in S} p(s)v(s) \geq U(q) = \sum_{s \in S} q(s)v(s).
\]
Sketch of the proof.

Let $M$ and $m$ be a best and a worst certain lotteries in $L(S)$. When $M \sim m$, choosing $v(s) = 0$ for all $s$ we have $\sum_{s \in S} p(s)v(s) = 0$ for all $p \in L(S)$. Consider the case that $M \succ m$. By $I$ and $C$, there must be a single number $v(s) \in [0, 1]$ such that

$$v(s) \circ M \oplus (1 - v(s)) \circ m \sim [s]$$

where $[s]$ is a certain lottery with prize $s$, i.e., $[s] = 1 \circ s$. In particular, $v(M) = 1$ and $v(m) = 0$. $I$ implies that

$$p \sim \left(\sum_{s \in S} p(s)v(s)\right) \circ M \oplus (1 - \sum_{s \in S} p(s)v(s)) \circ m.$$ 

Since $M \succ m$, we can show that

$$p \succeq q \iff \sum_{s \in S} p(s)v(s) \geq \sum_{s \in S} q(s)v(s).$$

\[\square\]
Note the function $U$ is a utility function representing the preferences on $L(S)$ while $v$ is a utility function defined over $S$, which is the building block for the construction of $U(p)$. We refer to $v$ as a vNM (Von Neumann-Morgenstern) utility function.

**Q** How can we construct the vNM utility function?

Let $s_i (\in S), i = 1, \ldots, K$ be a set of consequences and $s_1, s_K$ be the best and the worst consequences. That is, for any $i$,

$$[s_1] \succ [s_i] \succ [s_K].$$

Then, construct a function $v : S \rightarrow [0, 1]$ in the following way:

$v(s_1) = 1$ and $v(s_K) = 0$, and

$$[s_j] \sim v(s_j) \circ [s_1] \oplus (1 - v(s_j) \circ [s_K] \text{ for all } j.$$ 

By continuity axiom, we can find a unique value of $v(s_j) \in [0, 1]$. 

To what extent, vNM utility function is unique?

The vNM utilities are unique up to positive affine transformation (アフィン変換), i.e., multiplication by a positive number and adding any scalar, and are not invariant to arbitrary monotonic transformation (単調変換).

**Theorem 4**

Suppose $\succsim$ is a preference relation defined over $L(S)$ and let $v(s)$ be the vNM utilities representing the preference relation. Then, defining $w(s) = \alpha v(s) + \beta$ for all $s$ (for some $\alpha > 0$ and some $\beta$), the utility function $W(p) = \sum_{s \in S} p(s)w(s)$ also represents $\succsim$. 
Proof.

For any lotteries \( p, q \in L(S) \), \( p \succeq q \) if and only if

\[
\sum_{s \in S} p(s)v(s) \geq \sum_{s \in S} q(s)v(s).
\]

Now, the followings hold.

\[
\sum_{s \in S} p(s)w(s) = \sum_{s \in S} p(s)(\alpha v(s) + \beta) = \alpha \sum_{s \in S} p(s)v(s) + \beta.
\]

\[
\sum_{s \in S} q(s)w(s) = \sum_{s \in S} q(s)(\alpha v(s) + \beta) = \alpha \sum_{s \in S} q(s)v(s) + \beta.
\]

Thus,

\[
\sum_{s \in S} p(s)v(s) \geq \sum_{s \in S} q(s)v(s)
\]

holds if and only if

\[
\sum_{s \in S} p(s)w(s) \geq \sum_{s \in S} q(s)w(s) \quad \text{(for } \alpha > 0\text{).}
\]
Allais Paradox | アレのパラドックス (1)

Many experiments reveal systematic deviations from vNM assumptions. The most famous one is the **Allais paradox** (アレのパラドックス).

**Ex** Allais paradox

Choose first the between

\[ L_1 = [3000] \text{ and } L_2 = 0.8 \circ [4000] \oplus 0.2 \circ [0] \]

and then choose between

\[ L_3 = 0.5 \circ [3000] \oplus 0.5 \circ [0] \text{ and } L_4 = 0.4 \circ [4000] \oplus 0.6 \circ [0]. \]

Note that \( L_3 = 0.5 \circ L_1 \oplus 0.5 \circ [0] \text{ and } L_4 = 0.5 \circ L_3 \oplus 0.5 \circ [0]. \) Axiom \( I \) requires that the preference between \( L_1 \) and \( L_2 \) be the same as that between \( L_3 \) and \( L_4 \). However, a majority of people express the preferences \( L_1 \succ L_2 \) and \( L_3 \prec L_4 \), violating the axiom.
Assume $L_1 \succ L_2$ but $\alpha \circ L \oplus (1 - \alpha) \circ L_1 \prec \alpha \circ L \oplus (1 - \alpha) \circ L_2$. (In our example of Allais paradox, $\alpha = 0.5$ and $L = [0]$.)

Then, we can perform the following trick on the decision maker:

1. Take $\alpha \circ L \oplus (1 - \alpha) \circ L_1$.
2. Take instead $\alpha \circ L \oplus (1 - \alpha) \circ L_2$, which you prefer (and you pay me something...).
3. Let us agree to replace $L_2$ with $L_1$ in case $L_2$ realizes (and you pay me something now...).
4. Note that you hold $\alpha \circ L \oplus (1 - \alpha) \circ L_1$.
5. Let us start from the beginning...

This argument may make the independence axiom looks somewhat reasonable (and Allais paradox unreasonable).
Allais paradox can be viewed as a violation of independence axiom. The following paradox also shows that many people do not necessarily follow the expected utility maximization behavior.

Ex Zeckhauser’s paradox

Some bullets are loaded into a revolver with six chambers. The cylinder is then spun and the gun pointed at your head. Would you be prepared to pay more to get one bullet removed when only one bullet was loaded, or when four bullets were loaded?

Q People usually say they would pay more in the first case, because they would then be buying their lives for certain. Is this decision reasonable?

Rm Note that you cannot use your money once you die...
Suppose $X$ (resp. $Y$) is the most that you are willing to pay to get one bullet removed from a gun containing one (resp. four) bullet. Let $L$ mean death, and $W$ mean being alive after paying nothing. Let $C$ mean being alive after paying $X$, and $D$ mean being alive after paying $Y$. Note that

$$u(D) < u(C) \Leftrightarrow D \prec C \Leftrightarrow X < Y.$$ 

Let $u(L) = 0$ and $u(W) = 1$. Then,

$$u(C) = \frac{1}{6}u(L) + \frac{5}{6}u(W) = \frac{5}{6},$$

and

$$\frac{1}{2}u(L) + \frac{1}{2}u(D) = \frac{2}{3}u(L) + \frac{1}{3}u(W) \Rightarrow u(D) = \frac{2}{3}.$$ 

Since $u(D) < u(C)$, you must be ready to pay less to get one bullet removed when only one bullet was loaded than when four bullets were loaded.
We consider an agent’s “behavior” as a hypothetical response to the following questionnaire, one for each $A \subseteq X$:

$Q(A)$ Assume you must choose from a set of alternatives $A$. Which alternative do you choose?

- A choice function $C$ assigns to each set $A \subseteq X$ a unique element of $A$ with the interpretation that $C(A)$ is the chosen element from the set $A$.

Rm Here are a couple of remarks on choice functions:

1. We assume that the agent selects a *unique* element in $A$ for every question $Q(A)$.
2. The choice function $C$ does not need to be observable.
3. The agent behaving in accordance with $C$ will choose $C(A)$ if she has to make a choice from a set $A$. 
To guarantee the existence of a utility representation over consumption set, i.e., an infinite subset of $\mathbb{R}^n$, we need some additional axiom.

**Definition 7**

A preference relation $\succsim$ on $X$ is **continuous** (連続, Axiom 3) if $\{x^n\}$ (a sequence of consumption bundles) with limit $x$ satisfies the following two conditions for all $y \in X$.

1. if $x \succ y$, then for all $n$ sufficiently large, $x^n \succ y$, and
2. if $y \succ x$, then for all $n$ sufficiently large, $y \succ x^n$.

The equivalent definition of continuity is that the “at least as good as” and “no better than” sets for each point $x \in X$ are closed. This axiom rules out certain discontinuous (不連続な) behavior and guarantees that sudden preference reversals do not occur.

(Fg) Figures 1.2 and 1.3 (see JR, pp.9)
Given axioms 1-3, we can establish the existence of the (continuous) utility function.

**Theorem 5**

*Assume that $X$ is a convex subset of $\mathbb{R}^n$. If $\succeq$ is a continuous preference relation on $X$, then $\succeq$ is represented by a continuous utility function.*

Here are two remarks on continuity.

1. If $\succeq$ on $X$ is represented by a continuous function $U$, then $\succeq$ must be continuous.

2. The *lexicographic preferences* (辞書的選好) are not continuous.

**Theorem 6**

*The lexicographic preference relation $\succeq_L$ on $[0, 1] \times [0, 1]$, i.e., $(a_1, a_2) \succeq_L (b_1, b_2)$ if $a_1 > b_1$ or both $a_1 = b_1$ and $a_2 \geq b_2$, does not have a utility representation.*
Theorem 7

*If* $\succsim$ *is a continuous preference relation, then all consumer problems have a solution.*

Proof.

Since the budget set is convex, we can apply the first theorem in the previous slide to establish that the preferences are represented by a continuous utility function.

Then, by the *Weielstrass theorem* (ワイエルシュトラスの定理), there exists a maximum (and minimum) value of continuous functions if the domain is a compact (that is, closed and bounded) set and a range is $\mathbb{R}$. Since every budget set is compact and a utility function is continuous, there must exist a consumption bundle which gives a maximum utility value, a solution of the consumer problem.
We (Economics) assume that when the agent has in mind a preference relation $\succeq$ on $X$, then given any choice problem $Q(A)$ for $A \subseteq X$, she chooses an element in $A$ which is “$\succeq$-optimal.”

**Definition 8**

An **induced choice function** $C_\succeq$ is the function that assigns every nonempty set $A \subseteq X$ the $\succeq$-best element of $A$.

A choice function $C$ can be **rationalized** (合理化される) if there is a preference relation $\succeq$ on $X$ so that $C = C_\succeq$, i.e., $C(A) = C_\succeq(A)$ for any $A$ in the domain of $C$.

Q Under what conditions any choice functions can be presented “as if (あたかも)” derived from some preference relation?

**Definition 9**

Choice function $C$ satisfies **condition** $\alpha$ if for any $A \subset B$, $C(B) \in A$ implies $C(A) = C(B)$.
Theorem 8

Assume $C$ is a choice function with a domain containing at least all subsets of $X$ of size 2 or 3. If $C$ satisfies condition $\alpha$, then there is a preference relation $\succeq$ on $X$ so that $C = C_{\succeq}$.

Proof.

Define $\succeq$ by $x \succeq y$ if $x = C(\{x, y\})$. Let us first show that $\succeq$ satisfies completeness and transitivity.

Completeness: Follows from that $C = (\{x, y\})$ is well-defined.

Transitivity: If $x \succeq y$ and $y \succeq z$, then by definition of $\succeq$ we have $C(\{x, y\}) = x$ and $C(\{y, z\}) = y$. If $C(\{x, z\}) = z$, then, by condition $\alpha$, $C(\{x, y, z\}) \neq x$. Similarly, by $C(\{x, y\}) = x$ and condition $\alpha$, $C(\{x, y, z\}) \neq y$, and by $C(\{y, z\}) = y$ and condition $\alpha$, $C(\{x, y, z\}) \neq z$. A contradiction to $C(\{x, y, z\}) \in \{x, y, z\}$.

Next we show that $C(A) = C_{\succeq}(A)$ for all $A \subseteq X$. Suppose on contrary $C(A) \neq C_{\succeq}(A)$. That is, $C(A) = x$ and $C_{\succeq}(A) = y(\neq x)$. By definition of $\succeq$ and $y \succeq x$, this means $C(\{x, y\}) = y$, contradicting condition $\alpha$. □